Transportation Problem

Introduction:

The transportation problem is a particular type of LPP and can be regarded as a generalization of **assignment** problem.
A transportation problem can be described as follows:
Suppose that the factories F_i (i = 1, 2, 3, ..., m), called the **origins or sources** produce the non – negative quantities a_i (i = 1, 2, 3, ..., m) of a product and the non – negative quantities b_j (j = 1, 2, 3, ..., n) of the same product are required at the other n places, called the **destinations**, such that total quantity produced is equal to the total quantity required i.e.,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \text{..........} \quad \text{....(1)}$$

Also consider that c_{ij} is the cost of transportation of a unit from the i^{th} source to the j^{th} destination. Then the problem is to determine x_{ij}, the quantity transported from the i^{th} source to j^{th} destination to minimize the total transportation cost. i.e.,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \text{ is minimized.}$$

The transportation problem further can be described in tabular form as follows:

Destinations → **Sources** ↓	W_1	W_2	.....	W_j	.....	W_n	**Capacities of the Sources**
F_1	c_{11}	c_{12}	.....	c_{1j}	.....	c_{1n}	a_1
F_2	c_{21}	c_{22}	.....	c_{2j}	.....	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
F_i	c_{i1}	c_{i2}	.....	c_{ij}	.....	c_{in}	a_i
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
F_m	c_{m1}	c_{m2}	.....	c_{mj}	.....	c_{mn}	a_m
Requirements →	b_1	b_2	.....	b_j	.....	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The calculations are made directly on the transportation array given below which gives the current trial solution:

Destinations → **Sources ↓**	W_1	W_2	.....	W_j	.....	W_n	**Capacities of the Sources**
F_1	x_{11}	x_{12}	.....	x_{1j}	.....	x_{1n}	a_1
F_2	x_{21}	x_{22}	.....	x_{2j}	.....	x_{2n}	a_2
⋮	⋮	⋮		⋮		⋮	⋮
F_i	x_{i1}	x_{i2}	.....	x_{ij}	.....	x_{in}	a_i
⋮	⋮	⋮		⋮		⋮	⋮
F_m	x_{m1}	x_{m2}	.....	x_{mj}	.....	x_{mn}	a_m
Requirements →	b_1	b_2	.....	b_j	.....	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The above two tables can be combined together by writing the cost and cij within the bracket ( ) as follows:

Destinations → / Sources ↓	W_1	W_2	.....	W_j	.....	W_n	Capacities of the Sources
F_1	x_{11} (c_{11})	x_{12} (c_{12})	.....	x_{1j} (c_{1j})	.....	x_{1n} c_{1n}	a_1
F_2	x_{21} (c_{21})	x_{22} (c_{22})	.....	x_{2j} (c_{2j})	.....	$x_{2n}(c_{2n})$	a_2
⋮	⋮	⋮		⋮		⋮	⋮
F_i	x_{i1} (c_{i1})	x_{i2} (c_{i2})	.....	x_{ij} (c_{ij})	.....	$x_{in}(c_{in})$	a_i
⋮	⋮	⋮		⋮		⋮	⋮
F_m	x_{m1} (c_{m1})	x_{m2} (c_{m2})	.....	$x_{mj}(c_{mj})$	.....	$x_{mn}(c_{mn})$	a_m
Requirements →	b_1	b_2	.....	b_j	.....	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Few Important Definitions:

A Feasible Solution (A.F.S):

A feasible solution to a transportation problem is a set of non – negative individual allocation ($x_{ij} \geq 0$) which satisfies the row and column restrictions.

Basic Feasible Solution (B.F.S):

A feasible solution of m by n transportation problem is said to be a basic feasible solution if the total number of positive allocations x_{ij} is exactly equal to m + n – 1; i.e., one less than the sum of the number of rows and columns.

An Optimal Solution:

A feasible solution(not necessarily feasible) is said to be optimal if it minimizes the total transportation cost.

Non – degenerate Basic Feasible Solution:

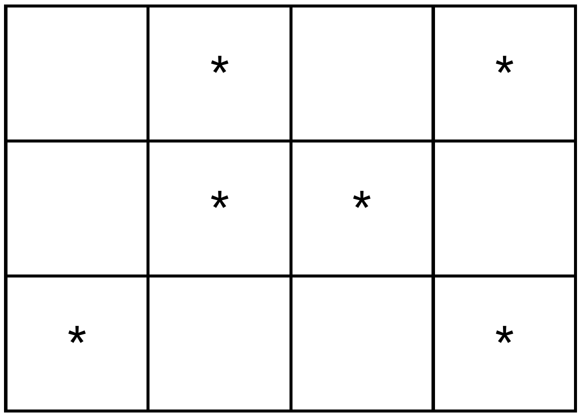
A feasible solution to a m by n transportation problem is said to be non – degenerate B.F.S. If

1. Total number of positive allocations is exactly equal to (m + n - 1).
2. These allocations are at independent positions.

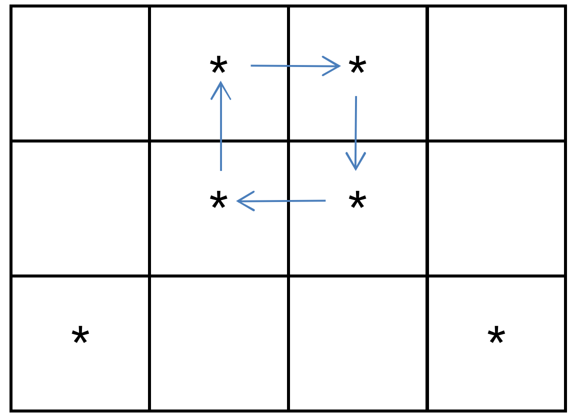
Otherwise degenerate.

Here by independent positions of the allocation we mean that it is always impossible to form an closed circuit (loop) by joining these allocations by horizontal and vertical lines only.

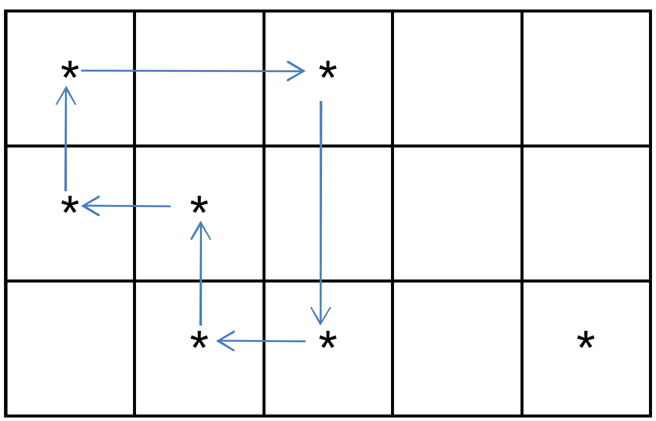
Independent positions



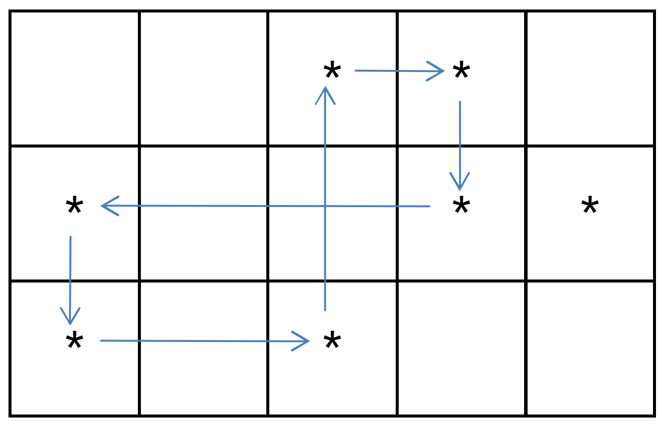
Non -Independent positions



Non - Independent positions



Non - Independent positions



Balanced and Unbalanced Transportation Problem:

A transportation problem in which $\sum a_i = \sum b_j$, is called a balanced transportation problem. Otherwise unbalanced transportation problem.

Solution to a transportation problem:

The solution (optimal) of a transportation problem consist of the following few steps:

1. To find the initial B.F.S.
2. To obtain an optimal solution by making successive improvements to the initial basic feasible solution (obtained in step 1) until no further decrease in the transportation cost is possible.

Method 1. North – west corner rule:

CL 1. Find the initial B.F.S. for the following transportation problem:

To

From

	W_1	W_2	W_3	**Supply**
F_1	2	7	4	**5**
F_2	3	3	1	**8**
F_3	5	4	7	**7**
F_4	1	6	2	**14**
Demand	**7**	**9**	**18**	**34**

To

From

	W_1	W_2	W_3	**Supply**
F_1	2 (5)	7	4	**5**
F_2	3 (2)	3 (6)	1	**8**
F_3	5	4 (3)	7 (4)	**7**
F_4	1	6	2 (14)	**14**
Demand	**7**	**9**	**18**	**34**

Total transportation cost is given by:

T = 2 × 5 + 3 × 2 + 3 × 6 + 4 × 3 + 7 × 4 + 2 × 14 = Rs 102.

Method 2. Lowest Cost Entry Method (Matrix Minima Method):

CL 1. Find the initial B.F.S. for the following transportation problem:

To

From		W_1	W_2	W_3	**Supply**
	F_1	2	7	4	**5**
	F_2	3	3	1	**8**
	F_3	5	4	7	**7**
	F_4	1	6	2	**14**
	Demand	**7**	**9**	**18**	**34**

To

From

	W_1	W_2	W_3	**Supply**
F_1	2	7 (2)	4 (3)	**5**
F_2	3	3	1 (8)	**8**
F_3	5	4 (7)	7	**7**
F_4	1 (7)	6	2 (7)	**14**
Demand	**7**	**9**	**18**	**34**

Total transportation cost is given by:

T = 2 × 7 + 3 × 4 + 8 × 1 + 7 × 4 + 7 × 1 + 7 × 2 = Rs 83.

Method 3. Vogal's Approximation Method (Unit Cost penalty Method):

CL 1. Find the initial B.F.S. for the following transportation problem:

To

From		W_1	W_2	W_3	**Supply**
	F_1	2	7	4	**5**
	F_2	3	3	1	**8**
	F_3	5	4	7	**7**
	F_4	1	6	2	**14**
	Demand	**7**	**9**	**18**	**34**

To

From

	W_1	W_2	W_3	Available	P_1	P_2	P_3	P_4	P_5	P_6	P_7
F_1	2 (5)	7	4	~~5~~	2	×	×	×	×	×	×
F_2	3	3	1 (8)	~~8~~	2	2	×	×	×	×	×
F_3	5	4 (7)	7	~~7~~	1	1	1	1	4	4	×
F_4	1 (2)	6 (2)	2 (10)	~~14~~	1	1	1	5	6	×	×
Demand	~~7~~	~~9~~	~~18~~	**34**							

	W_1	W_2	W_3
P_1	1	1	1
P_2	2	1	1
P_3	4	2	5
P_4	4	2	×
P_5	×	2	×
P_6	×	4	×
P_7	×	×	×

Total transportation cost is given by:

T = 5 × 2 + 8 × 1 + 7 × 4 + 2 × 1 + 2 × 6 + 10 × 2 = Rs 80.

Optimality Test:

After getting the initial F.S. of a transportation problem, we test this solution for optimality i.e., we check whether the feasible solution obtained, minimizes the total transportation cost or not. Thus we start the optimality test to a F.S, consisting of (m + n - 1) allocation in independent positions i.e., to a non – degenerate B.F.S.

In general there are following two methods used for the test of optimality of the solution.

(i) The Stepping – stone method

(ii) The modified distribution (MODI) Method or u – v method.

The Stepping Stone Method:

Consider the matrix giving the first feasible solution. To test the optimality of the solution, we start with an empty cell (i.e., a cell in which there is no allocation) and allocate +1 unit. In order to maintain the row and column sum unchanged we make necessary adjustment to the solution. The net change in total cost resulting from this adjustment, is calculated, it is called the evaluation for the empty set. If this cell evaluation is positive, the adjustment would increase the total cost, if it is negative, it would decrease the total cost. Since there are mn – (m + n - 1) = (m - 1) (n - 1) empty cells, therefore there are (m - 1) and (n - 1) such cell evaluations.

If all the cell evaluations are positive or zero, then we can not decrease the total transportation cost and hence the required solution under test is the required optimal solution.

The problem of computing cell evaluation for all unoccupied cells individually is very complicated.

Computational Procedure of Optimality Test:

After getting the initial B.F.S. of a transportation problem, we test solution for optimality as follows:

1. For a B.F.S. we determine a set of (m + n) numbers

   u_i, i = 1, 2,....., m

   v_j, j = 1, 2,....., n

Such that for occupied cell (r, s)

   $c_{rs} = u_r + v_r$

For this we assign an arbitrary value to one of the u_i's or v_j's then the rest (m + n - 1) of them can easily be solved algebraically from the relation $c_{rs} = u_r + v_r$ for occupied cells. Generally we chose that u_i or $v_j = 0$ for which the corresponding row or column have the maximum number of individual allocations.

CL 2: A company has four plants P_1, P_2, P_3, P_4, form which it supplies to three markets M_1, M_2, M_3. Determine the optimal transportation plan from the following data giving the plant to market shifting cost, quantities available at each plant and quantities required at each market.

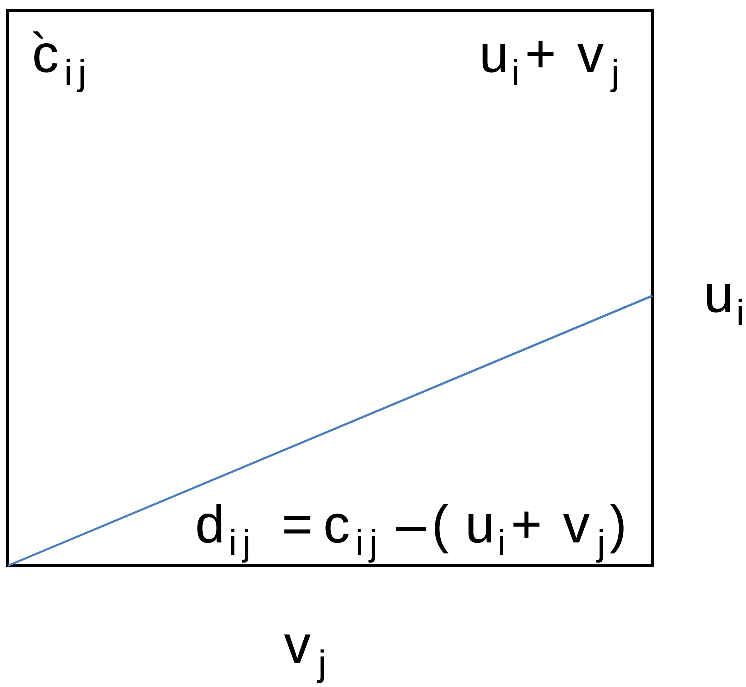
Market	Plant				Required
	P_1	P_2	P_3	P_4	
M_1	19	14	23	11	11
M_2	15	16	12	21	13
M_3	30	25	16	39	19
Available	6	10	12	15	43

				a_i
19	14	23	11 (11)	11
15 (6)	16 (3)	12	21 (4)	13
30	25 (7)	16 (12)	39	19
b_j: 6	10	12	15	

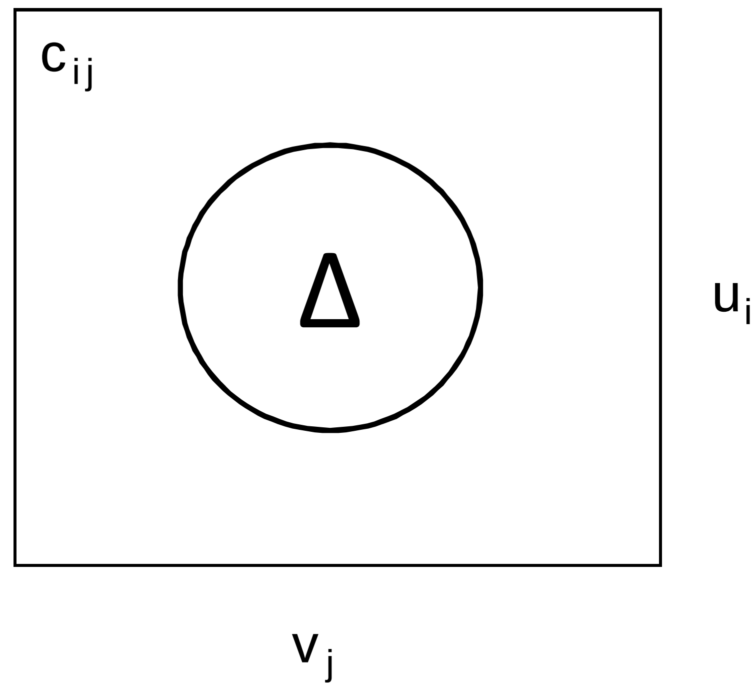
Total transportation cost is given by:

T = 6 × 15 + 3 × 16 + 7 × 25 + 12 × 16 + 11 × 11 + 4 × 21 = Rs 710.

Empty Cells:



Non-empty Cells:



$$c_{ij} = u_i + v_j$$

				a_i
19 5 14	14 6 8	23 -3 26	11 (11)	$u_1 = -10$
15 (6)	16 (3)	12 7 5	21 (4)	$u_2 = 0$
30 24 6	25 (7)	16 (12)	39 30 9	$u_3 = 9$
$v_1 = 15$	$v_2 = 16$	$v_3 = 7$	$v_4 = 21$	

Since all the dij are ≥ 0. Therefore the transportation cost = Rs 710.

CL 1: A company has three plants 1, 2, 3, form which it supplies to 4 markets 1, 2, 3. Determine the optimal transportation plan from the following data giving the plant to market shifting cost, quantities available at each plant and quantities required at each market.

Market	Plant			Required
	1	2	3	
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Available	7	9	18	34

Solution: The initial B.F.S. of the above problem (by Vogel's method) is given in the following table.

Total transportation cost

$$= 5 \times 2 + 2 \times 1 + 7 \times 4 + 2 \times 6 + 8 \times 1 + 10 \times 2 = \text{Rs } 80$$

				a_i
	2 (5)	7	4	5
	3	3	1 (8)	8
	5	4 (7)	7	7
	1 (2)	6 (2)	2 (10)	14
b_j	7	9	18	

Since all $d_{ij} \not\geq 0$, therefore the solution is not optimal

				u_i
	2 (5)	7 7 0	4 3 1	$u_1 = 1$
	3 0 3	3 5 (+ 2) -2	1 (8) (- 2)	$u_2 = -1$
	5 -1 6	4 (7)	7 0 7	$u_3 = -2$
	1 (2)	6 (- 2) (2)	2 (10) (+ 2)	$u_4 = 0$
v_j	$v_1 = 1$	$v_2 = 6$	$v_3 = 2$	

Since all $d_{ij} \geq 0$, therefore the solution is optimal.

				u_i
	2 (5)	7 5 2	4 3 1	$u_1 = 1$
	3 0 3	3 5 (2)	1 (6)	$u_2 = -1$
	5 1 4	4 (7)	7 2 5	$u_3 = 0$
	1 (2)	6 4 2	2 (12)	$u_4 = 0$
v_j	$v_1 = 1$	$v_2 = 4$	$v_3 = 2$	

Therefore,

From source 1 transport 5 units to destination 1.

From source 2 transport 2 and 6 units to destinations 2 and 3 respectively.

From source 3 transport 7 units to destination 2.

From source 4 transport 2 and 12 units to destinations 1 and 3 respectively.

And , the total transportation cost = Rs 76

CL 3: Solve the following transportation problem and test for optimality

Market	Plant				Required
	S_1	S_2	S_3	S_4	
D_1	21	16	25	13	11
D_2	17	18	14	23	13
D_3	32	27	18	41	19
Available	6	10	12	15	43

Degeneracy in Transportation Problem:

In the transportation problem, degeneracy occurs whenever the number of independent individual allocation is less than (m + n - 1). Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

In such cases, to resolve degeneracy, we allocate an extremely small amount (close to zero) to one or empty cells of the matrix (generally lowest cost cells if possible), so that the total number of occupied (allocated) cells becomes (m + n - 1) at independent positions. We denote this small amount by Δ (delta) or $\in$ (epsilon) satisfying the following conditions.

1. $0 < \Delta\ x_{ij}$ for all $x_{ij} > 0$
2. $\Delta + 0 = \Delta = 0 + \Delta$
3. $x_{ij} \pm \Delta = x_{ij}$ for all $x_{ij} > 0$
4. If there are more than one Δ's introduced in the solution then,
    1. If Δ and Δ' are in the same row, $\Delta < \Delta'$ when Δ is to the left of Δ'.
    2. If Δ and Δ' are in the same column, $\Delta < \Delta'$ when Δ is above Δ'.

It is clear that after introducing Δ satisfying the above conditions, the original solution of the problem is not changed. It is merely a technique to apply the optimality test and is omitted ultimately.

CL 4: Solve the following transportation problem and test for optimality

To

From

	A	B	C	D	E	F	a_i
1	5	3	7	3	8	5	3
2	5	6	12	5	7	11	4
3	2	1	3	4	8	2	2
4	9	6	10	5	10	9	8
b_j	3	3	6	2	1	2	17

By V.A.M. The initial B F S is given by the following

							a_i
	5	3 (1)	7	3	8	5 (2)	3
	5 (3)	6	12	5 (Δ)	7 (1)	11	4
	2	1	3 (2)	4	8	2	2
	9	6 (2)	10 (4)	5 (2)	10	9	8
bj	3	3	6	2	1	2	

Since the total number of allocations is 8 which is one less than $m + n - 1 = 9$. Hence this is a degenerate solution. To remove we allocate the an amount Δ to the appropriate cell.

Now testing for optimality we get

						A_i
5 2 3	3 (1)	7 7 0	3 2 1	8 4 4	5 (2)	$u_2 = -3$
5 (3)	6 6 0	12 10 2	5 (Δ)	7 (1)	11 8 3	$u_2 = 0$
2 -2 4	1 -1 2	3 (2)	4 -2 6	8 0 8	2 1 1	$u_3 = -7$
9 5 4	6 (2)	10 (4)	5 (2)	10 7 3	9 8 1	$u_4 = 0$
bj $v_1 = 5$	$v_2 = 6$	$v_3 = 10$	$v_4 = 5$	$v_5 = 7$	$v_6 = 8$	

Since $d_{ij} \geq 0$ therefore the solution is optimal and is given by:

Therefore,

From source 1 transport 1 and 2 units to destinations 2 and 6 respectively.

From source 2 transport 3 and 1 unit to destinations 1 and 5 respectively.

From source 3 transport 2 units to destination 3.

From source 4 transport 2, 4 and 2 units to destinations 2, 3 and 4 respectively.

And , the total transportation cost = Rs 103.

Problems of Unbalanced Transportation Problem:

CL 5: Determine the optimal transportation plan from the following table given the plant to market shipping costs, and quantities required at each market and available at each plant:

Plant	Markets				Supply
	M_1	M_2	M_3	M_4	
F_1	11	20	7	8	50
F_2	21	16	10	12	40
F_3	8	12	18	9	70
Need	30	25	35	40	

Since the total availability at 3 plants is 30 more than the total requirements, hence this is a unbalanced transportation problem. To convert this problem in to a balance transportation problem we introduce the fictitious, market M_5 with the requirement 30 such that the cost of transportation from plants to this market are 0.

Therefore the balanced transportation problem is given by:

Plant	Markets					Supply
	M_1	M_2	M_3	M_4	M_5	
F_1	11	20	7	8	0	50
F_2	21	16	10	12	0	40
F_3	8	12	18	9	0	70
Need	30	25	35	40	30	

Solving the above problem by V A M is given by

Now testing for optimality we get

						A_i
	11 7 4	20 11 9	7 (25)	8 (25)	0 -3 3	$u_1 = -1$
	21 10 11	16 14 2	10 (10)	12 11 1	0 (30)	$u_2 = 2$
	8 (30)	12 (25)	18 8 10	9 (15)	0 -2 2	$u_3 = 0$
bj	$v_1 = 8$	$v_2 = 12$	$v_3 = 8$	$v_4 = 9$	$v_5 = -3$	

Since $d_{ij} \geq 0$ therefore the solution is optimal and is given by:

Since $d_{ij} \geq 0$ therefore the solution is optimal and is given by:

Transport from Plant F_1 to Market M_3 25 units
Transport from Plant F_1 to Market M_4 25 units
Transport from Plant F_2 to Market M_3 10 units
Transport from Plant F_3 to Market M_1 30 units
Transport from Plant F_3 to Market M_2 25 units
Transport from Plant F_3 to Market M_4 15 units

Total transportation cost = Rs 1150.

Prohibited Transportation Route:

Some times it is not possible to transport goods from certain sources to certain destination due to road blockade or for any reason, such type of problems can be handled by assigning a very large cost say M (or ∞) to that route or cell.

Case: Given the Following Data:

Destinations

Sources	1	2	3	Capacity
1	2	2	3	10
2	4	1	2	15
3	1	3	-	40
Demand	20	15	30	

The cost of shipment from third source to the third destination is not known. How many units should be transported from sources to the destinations so that the total cost of transporting all the units to their destinations is a minimum.

Solution:

Since the cost C33 in cell (3, 3) not known, so we assign a very large cost say M to this cell. By Vogel's approximation method an initial B F S is shown in the following table:

2 4 - M M - 2	2 6 - M M - 4	3 (10)	$u_1 = 3 - M$
4 3 - M M + 1	1 5 - M M - 4	2 (15)	$u_2 = 2 - M$
1 (20)	3 (15)	M (5)	$u_3 = 0$
$v_1 = 1$	$v_2 = 3$	$v_3 = M$	

Here all $d_{ij} \geq 0$, since M is very large. So this solution is optimal. And is known as pseudo optimum feasible solution.

		To										
From		**W_1**	**W_2**	**W_3**	**Available**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**	**P_6**	**P_7**
	F_1	2 (5)	7	4	~~5~~	2	×	×	×	×	×	×
	F_2	3	3	1 (8)	~~8~~	2	2	×	×	×	×	×
	F_3	5	4 (7)	7	**7**	1	1	1	1	4	4	×
	F_4	1 (2)	6 (2)	2 (10)	~~14~~	1	1	1	5	6	×	×
Demand		~~7~~	**9**	~~18~~	**34**							

	W_1	W_2	W_3
P_1	2	1	1
P_2	2	1	1
P_3	4	2	5
P_4	4	2	×
P_5	×	2	×
P_6	×	4	×
P_7	×	×	×