Assignment Problem

This is a special type of LPP in which the objective function is to find the optimum allocation of a number of tasks (jobs) to an equal number of facilities (persons). Here we make the assumption that each person can perform each job but with varying degree of efficiency. For example, a departmental head may have 4 persons available for assignment and 4 jobs to fill. Then his interest is to find the best assignment which will be in the best interest of the department.

General form of an Assignment Problem:

The assignment problem can be stated in the form of n × n, matrix $[c_{ij}]$ called the cost of effectiveness matrix, where c_{ij} is the cost of assigning i^{th} person (facility) to j^{th} job.

Effectiveness Matrix:

Jobs

Persons

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & \text{.........} & j & \text{.........} & n \\
\hline
1 & c_{11} & c_{12} & c_{13} & \text{.........} & c_{ij} & \text{.........} & c_{in} \\
2 & c_{21} & c_{22} & c_{23} & \text{.........} & c_{2j} & \text{.........} & c_{2n} \\
3 & c_{31} & c_{32} & c_{33} & \text{.........} & c_{3j} & \text{.........} & c_{3n} \\
\text{.......} & \text{.......} & \text{........} & \text{.........} & \text{........} & \text{.........} & \text{.......} & \text{........} \\
i & c_{i1} & c_{i2} & c_{i3} & \text{........} & c_{ij} & \text{........} & c_{in} \\
\text{......} & \text{........} & \text{.......} & \text{.........} & \text{..........} & \text{.......} & \text{.........} & \text{........} \\
n & c_{n1} & c_{n2} & c_{n3} & \text{.........} & c_{nj} & \text{........} & c_{nn}
\end{array}
$$

A person can be assigned to n jobs in n! Possible ways. One method may be to find all possible n! assignments and evaluate total cost in all cases. Then the assignment with minimum cost (as required) will give the optimal assignment. But this method is extremely laborious. For example if n = 8, then the number of such possible assignments is 8! = 40320. The evaluation cost for all these allocation will take huge time. Therefore there is a need to develop an easy computational technique for the solution of assignment problem.

Mathematical Formulation :

Mathematical formulation of an assignment problem can be stated as follows:

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{Where } x_{ij} = \begin{cases} 1, \text{ if } i^{th} \text{ person is assigned to the } j^{th} \text{ job.} \\ 0, \text{ if } i^{th} \text{ person is not assigned the } j^{th} \text{ job.} \end{cases}$$

Subject to the conditions:

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, ..., \quad n.$$

Which means that only one job is done by the i^{th} person, i=1,2,......,n.

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots\ldots., \quad n.$$

Which means that only one person should be assigned the j^{th} job, j=1,2,......,n.

Reduction Theorem:

If in an assignment problem, a constant is added or subtracted to every element of a row (or column) of the cost matrix $[c_{ij}]$, then an assignment which minimizes the total cost for one matrix, also minimizes the total cost for the other matrix.

Solution of an Assignment Problem:

(Hungarian Method or Reduced Matrix Method)

1. Solve the following minimal assignment problem:

Man →

Job ↓	1	2	3	4
1	12	30	21	15
2	18	33	9	31
3	44	25	24	21
4	23	30	28	14

Step 1: Select the minimum of each row and subtract from all the elements of corresponding row.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step 2: Select the minimum of each column and subtract from all the elements of corresponding column.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3: Starting with row 1 of the matrix obtained in step 2. Examine rows successively until a row with exactly one zero element is found. Mark (□) at this zero, as an assignment is made there. Mark (×) at all the other zeros in the column (in which we mark □) to show that they can not be used to make other assignments. Proceed in this way until the last row is examined.

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	0	3	$\xcancel{0}$
9	12	14	$\boxed{0}$

Step 4: After examining all the rows completely. Proceed similarly examining the columns. Examine column starting with column 1 until a column containing exactly one unmarked 0 is found. Mark (□) at this zero. Mark (×) at all the other zeros in the rows (in which we mark □). Proceed in this way until the last column is examined.

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	$\xcancel{0}$
9	12	14	$\boxed{0}$

Step 5. Continue the these operations until successively until we reach to any of the two situations:

a. All the zero's are marked □ or ×.

b. The remaining unmarked zeros lies at-least two in a row or a column.

In case (A) we have the maximal assignment (assignment as much as we can) and in case (b) still we have some zeros to be treated for which we use the trail and error method to avoid the use of highly complicated algorithm.

Now there are two possibilities:

a. If it has an assignment in each row and each column (i.e. The total number of marked □ zero is exactly equal to n), then the complete optimal solution is obtained.

Since every row and every column has one assignment, so we have the complete optimal zero assignment.

JOB	1	2	3	4
MAN	1	3	2	4

Which is the optimal assignment.

b. If it does not contain assignment in every row and every column (i.e. The total number of marked □ zeros is less than n), then one has to modify the cost (effectiveness) matrix by adding or subtracting to create some more zeros to it.

2. A department head has 4 subordinates, and 4 tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate of the times each man would take to perform each task is given in the effectiveness matrix below. How should the tasks to be allocated, one to a man, so as to minimize the total man hour?

Man →

Job ↓	1	2	3	4
1	2	3	4	5
2	4	5	6	7
3	7	8	9	8
4	3	5	8	4

$\xcancel{0}$	$\boxed{0}$	$\xcancel{0}$	2
$\xcancel{0}$	$\xcancel{0}$	$\boxed{0}$	2
$\xcancel{0}$	$\xcancel{0}$	$\xcancel{0}$	$\boxed{0}$
$\boxed{0}$	1	3	$\xcancel{0}$

Since every row and every column has one assignment, so we have the complete optimal zero assignment.

JOB	1	2	3	4
MAN	2	3	4	1

Which is the optimal assignment.

2. Solve the following assignment problem?

Man → Job ↓	1	2	3	4	5	6
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

$\xcancel{0}$	13	49	$\boxed{0}$	$\xcancel{0}$	13
$\xcancel{0}$	35	29	5	10	$\boxed{0}$
13	$\xcancel{0}$	63	7	7	$\xcancel{0}$
47	15	$\boxed{0}$	20	2	$\xcancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

1. We mark (√) row 3 in which there is no assignment.

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13		
$\cancel{0}$	35	29	5	10	$\boxed{0}$		
13	$\cancel{0}$	63	7	7	$\cancel{0}$	√	①
47	15	$\boxed{0}$	20	2	$\cancel{0}$		
25	$\boxed{0}$	46	9	4	2		
$\boxed{0}$	53	50	26	4	20		

2. We mark (√) columns 2 and 6 in which have zeros in marked rows.

$$\begin{array}{|cccccc|cc}
\cline{1-6}
\xcancel{0} & 13 & 49 & \boxed{0} & \xcancel{0} & 13 & & \\
\xcancel{0} & 35 & 29 & 5 & 10 & \boxed{0} & & \\
13 & \xcancel{0} & 63 & 7 & 7 & \xcancel{0} & \surd & \textcircled{1} \\
47 & 15 & \boxed{0} & 20 & 2 & \xcancel{0} & & \\
25 & \boxed{0} & 46 & 9 & 4 & 2 & & \\
\boxed{0} & 53 & 50 & 26 & 4 & 20 & & \\
\cline{1-6}
\multicolumn{1}{c}{} & \surd & & & & \surd & & \\
\multicolumn{1}{c}{} & \textcircled{2} & & & & \textcircled{3} & &
\end{array}$$

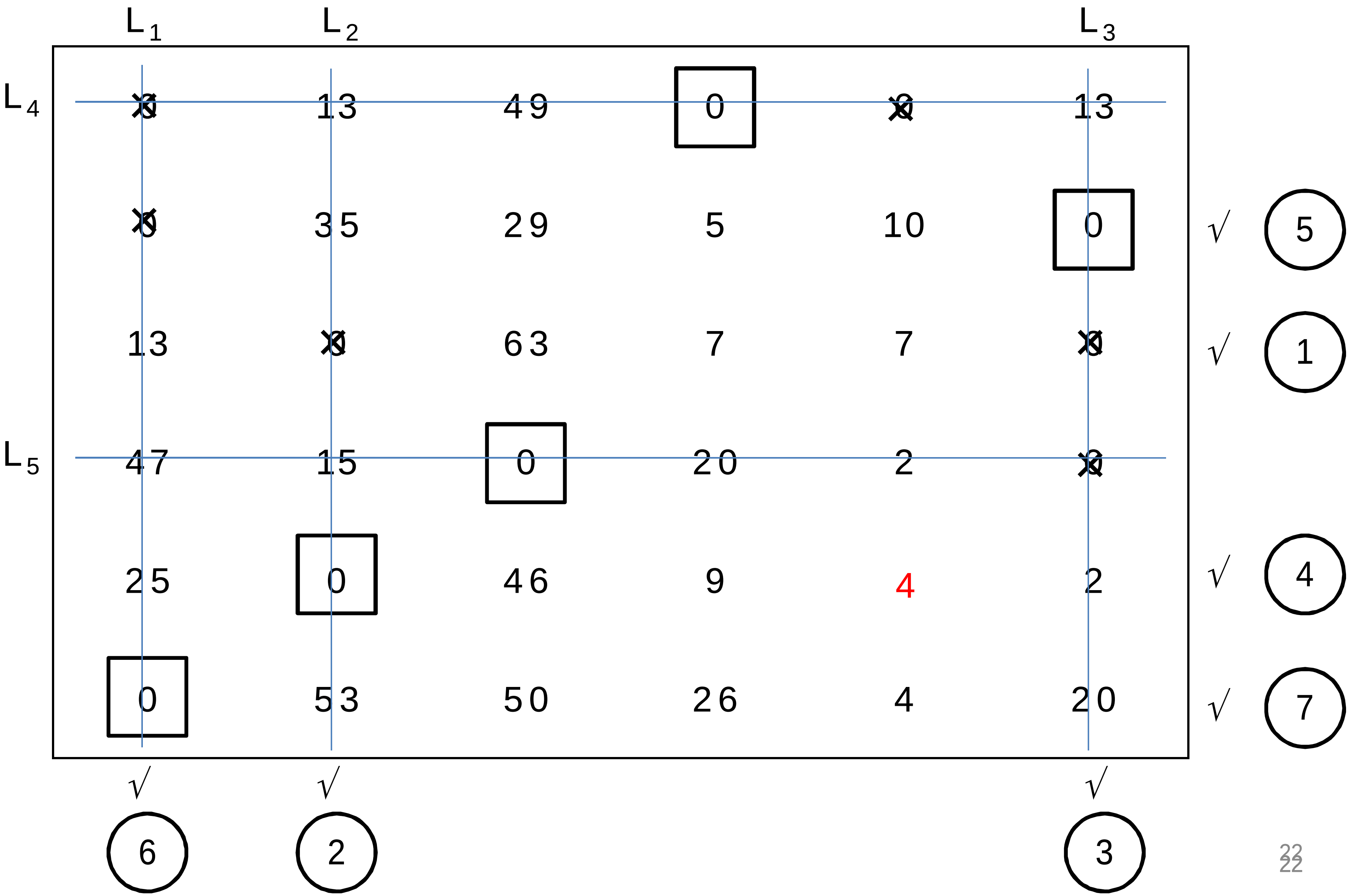
3. We mark (√) rows 5 and 2 which have assignments in marked columns.

$$\begin{array}{|cccccc|cc}
\hline
\xcancel{0} & 13 & 49 & \boxed{0} & \xcancel{0} & 13 & & \\
\xcancel{0} & 35 & 29 & 5 & 10 & \boxed{0} & \surd & \textcircled{5} \\
13 & \xcancel{0} & 63 & 7 & 7 & \xcancel{0} & \surd & \textcircled{1} \\
47 & 15 & \boxed{0} & 20 & 2 & \xcancel{0} & & \\
25 & \boxed{0} & 46 & 9 & 4 & 2 & \surd & \textcircled{4} \\
\boxed{0} & 53 & 50 & 26 & 4 & 20 & & \\
\hline
 & \surd & & & & \surd & & \\
 & \textcircled{2} & & & & \textcircled{3} & &
\end{array}$$

4. We mark (√) column 1 (not already marked) which has 0 in the marked row 2.

$\xcancel{0}$	13	49	$\boxed{0}$	$\xcancel{0}$	13		
$\xcancel{0}$	35	29	5	10	$\boxed{0}$	√	⑤
13	$\xcancel{0}$	63	7	7	$\xcancel{0}$	√	①
47	15	$\boxed{0}$	20	2	$\xcancel{0}$		
25	$\boxed{0}$	46	9	4	2	√	④
$\boxed{0}$	53	50	26	4	20	√	⑦
√	√				√		
⑥	②				③		

5. Now we draw lines through all marked columns 1, 2, 6. Then we draw lines through unmarked row 1 and 4 having zeros through which there is no line. In this way we get the minimum number of lines covering all zeros.



	L_1	L_2				L_3		
L_4	0	13	49	0	0	13		
	0	35	29	5	10	0	√	⑤
	13	0	63	7	7	0	√	①
L_5	47	15	0	20	2	0		
	25	0	46	9	4	2	√	④
	0	53	50	26	4	20	√	⑦
	√	√				√		
	⑥	②				③		

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6. Now select minimum of the left out elements. Subtract it from the left out elements, add at the intersection of lines. While the elements lying on the lines will remain as they were.

4	17	49	$\boxed{0}$	$\xcancel{0}$	17
$\boxed{0}$	35	25	1	6	$\xcancel{0}$
13	$\xcancel{0}$	59	3	3	$\boxed{0}$
51	19	$\boxed{0}$	20	2	4
25	$\boxed{0}$	42	5	$\xcancel{0}$	2
$\xcancel{0}$	53	46	22	$\boxed{0}$	20

Solution to the assignment problem is given by:

Job	a	b	c	d	e	f
Man	4	1	6	3	2	5

2. Solve the following assignment problem?

Man → Job ↓	1	2	3	4	5
A	1	3	2	3	6
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4

Case: An air – line that operates seven days a week has time table shown below. Cruse must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller layover.

Delhi - Jaipur			Jaipur - Delhi		
Fligth No	Departure	Arrival	Flight No	Departure	Arrival
1	7.00 am	8.00 am	101	8.00 am	9.15 am
2	8.00 am	9.00 am	102	8.30 am	9.45 am
3	1.30 pm	2.30 pm	103	12.00 noon	1.15 pm
4	6.30 pm	7.30 pm	104	5.30 pm	6.45 pm

For each pair also mention the town where the crew should be based.

Solution:

Layover time in hours when crew based at delhi.

Flight	101	102	103	104
1	24	24.5	28	9.5
2	23	23.5	27	8.5
3	17.5	18	21.5	27
4	12.5	13	16.5	22

Layover time in hours when crew based at Jaipur.

Flight	101	102	103	104
1	21.75	21.25	17.75	12.25
2	22.75	22.25	18.75	13.25
3	28.25	27.75	24.25	18.75
4	9.25	8.75	5.25	23.75

To avoid the fractions we consider either the layover times in terms of quarter hour as one unit of time or the layover times for 4 weeks. Thus multiplying the matrices by 4, the modified matrices are as follows:

Flight	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

And

Flight	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Now we combine the two tables, choosing that base which gives a lesser layover time for each pairing. The layover times marked with (*) denote the crew based at Jaipur, otherwise the crew is based at Delhi. Thus we get the following table.

Table 6. (Minimum Layover times table)

87*	85*	71*	38
91*	89*	75*	34
70	72	86	75*
37*	35*	21*	88

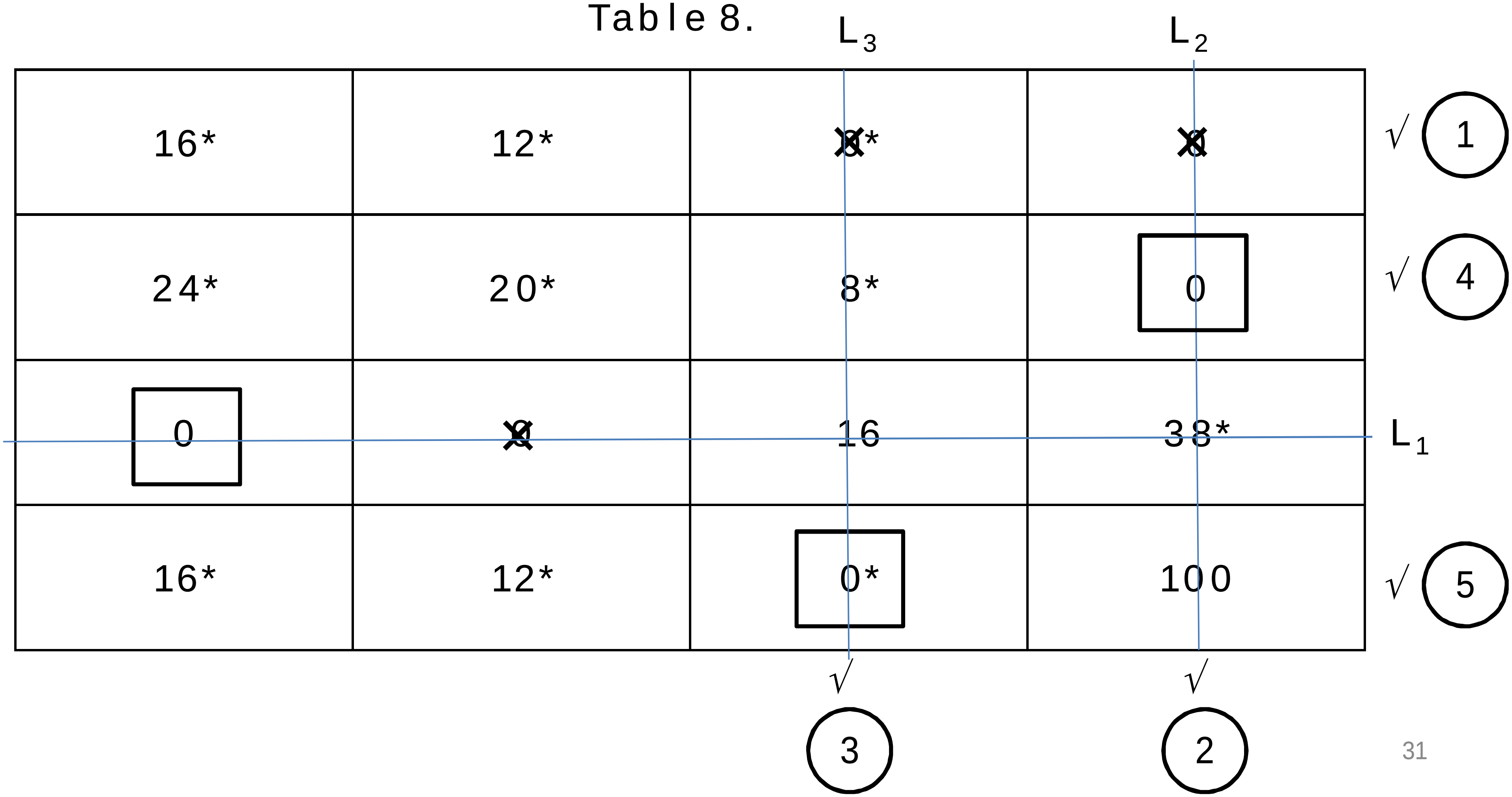
Subtracting the smallest element of each row from every element of the corresponding row and then subtracting the smallest element of each column from every element of the corresponding column, we get the following matrix.

Table 7.

			L_3	
49*	45*	33*	$\boxed{0}$	√ ③
57*	53*	41*	$\cancel{0}$	√ ①
$\boxed{0}$	$\cancel{0}$	16	5*	L_1
16*	12*	$\boxed{0*}$	67	L_2
			√	
			②	

Applying the modification operations and making assignments again we get the following table.

Table 8.



		L_3	L_2	
16*	12*	~~0~~*	~~0~~	√ ①
24*	20*	8*	$\boxed{0}$	√ ④
$\boxed{0}$	~~0~~	16	38*	L_1
16*	12*	$\boxed{0}$*	100	√ ⑤
		√ ③	√ ②	

Applying the modification operations and making assignments again we get the following table.

Table 9.

4*	$\boxed{0*}$	$\xcancel{0}$*	$\xcancel{0}$
12*	8*	8*	$\boxed{0}$
$\boxed{0}$	$\xcancel{0}$	28	50*
4*	$\xcancel{0}$*	$\boxed{0*}$	100

The optimal solution to the problem is given by following

1 → 102 (Crew at jaipur)
2 → 104 (crew at Jaipur)
3 → 101 (crew at Jaipur)
4 → 103 (crew at Delhi)

The minimum layover time is 210 hours for 4 weeks i.e. 52 hours and 30 minutes per week.

Case: A small air – plane company operating 7 days a week, serves three cities A, B and C according to the schedule shown in the following table. The layover cost per shop is roughly proportional to the square of the layover time. How should planes be assigned the flights so as to minimize the total layover cost?

Flight No.	**From**	**Departure**	**To**	**Arrival**
A1B	A	09.00 AM	B	Noon
A2B	A	10.00 AM	B	01.00 PM
A3B	A	03.00 PM	B	06.00 PM
A4C	A	08.00 PM	C	Midnight
A5C	A	10.00 PM	C	02.00 AM
B1A	B	04.00 AM	A	07.00 AM
B2A	B	11.00 AM	A	02.00 PM
B3A	B	03.00 PM	A	06.00 PM
C1A	C	07.00 AM	A	11.00 AM
C2A	C	03.00 PM	A	07.00 PM

Unbalanced Transportation Problem :

As assignment problems is called unbalanced assignment problem whenever the number of tasks (jobs) are not equal to the number of facilities (persons). Thus, the cost of matrix of an assignment problem is not the square matrix. For the solution of such problems we ass the dummy rows and dummy columns to the given matrix to make it a square matrix. The costs in these dummy rows or columns are taken to be 0. Now the problem reduce to the balanced assignment problem and can be solved by assignment algorithm.

Maximization Problem :

Sometimes the assignment problem deals with the maximization of the objective function i.e., the problem may be to assign persons to the jobs in such a way that the expected profit is maximized. Such maximization problem may be solved by converting it to minimization problem. This is done converting the profit matrix to the cost (i.e. loss) matrix in either of the following two ways.

(i) Subtract each element of the given matrix (profit matrix) from the greatest element of the matrix to get the equivalent cost (i.e. loss) matrix.

or

(ii) Place minus sign before each element of the profit matrix to get the equivalent cost matrix.

Case: Alpha corporation has four plants each of which can manufacture any of the four products. Production costs differ from plant to plant as do sales revenue. From the following data, obtain which product each plant should produce to maximize profit.

Sales revenue (Rs 1000)

		Product		
Plant	A	B	C	D
1	50	68	49	62
2	60	70	51	74
3	55	67	53	70
4	58	65	54	69

Production cost (Rs 1000)

		Product		
Plant	A	B	C	D
1	49	60	45	61
2	55	63	45	69
3	52	62	49	68
4	55	64	48	66

$\xcancel{0}$	13	49	$\boxed{0}$	$\xcancel{0}$	13
$\xcancel{0}$	35	29	5	10	$\boxed{0}$
13	$\xcancel{0}$	63	7	7	$\xcancel{0}$
47	15	$\boxed{0}$	20	2	$\xcancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

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