GAME THEORY

Introduction:

Game theory is concerned with a type of decision problem characterized by a conflict or competition among two or more competitors.

Union Leader and Management involve in collective bargaining

Political Negotiations

Promotional Decisions, etc.

The theory of games has started in 20th century. The mathematical treatment of games was developed, when John Von Neumann and Morgenstern published with their work “Theory of games and economic behaviour” in 1944. The approach to the competitive problems developed by J. Von Neumann (Known as father of game theory) utilizes the maximax principle which involve the fundamental idea of minimization of the maximum loss or maximization of minimum gain. The game theory is capable of analyzing very simple competitive situations, it can not handle all the competitive situations that may arise.

Game theory is a mathematical theory that deals with the general features of competitive situations. In many practical problems, it is required to take decision in a situation where there are two or more opposite parties with conflicting interests and the action of one depends upon the action which the opponent takes. Such a situation is termed as a **"competitive situation"**. A great variety of competitive situation is commonly seen in every day life e.g., in military battles, political campaigns, election advertising, and marketing campaigns etc.

In all the competitive situations, one may assume that each opponent is going to act in some rational manner and will attempt to resolve the conflict of interest in his favour.

Competitive Games:

A competitive situation is called a competitive game if it has the following properties.

1. There are finite number of competitors called players.
2. Every of the n – players has available to him a list of finite number of possible courses.
3. A play is said to be played when each of the players (competitors) chooses a single course of action from the list of courses of actions available to him.
4. Every play i.e. combination if courses of action determines an outcome (which may be money) which determines a set of payments (positive or negative, or zero) from one to each player.

Finite or Infinite Games:

Here we shall use the word move to mean a point in a game at which one of the players pick outs an alternative from some set of alternatives. Also the word ‘choice’ may be used to mean the alternative picked out. Ordinarily the word move is used for both notions.

A game said to be finite game if it has a finite number of moves, each of which involves only a finite number of alternatives.
A game which is not a finite game is called an **infinite game**.

Zero Sum Game:

Consider a game where there are n components, and competitor i has N_i courses of action available to him. Then the total number of possible outcomes to play of the game will be N_1, N_2, N_3, ... , N_n. Let a particular outcome θ result in a payment p (i, θ) to competitor i. Then the game is called a zero sum game, if for every possible outcome θ we have

$$\sum_{i=1}^{n} p(i, \theta) = 0$$

In other words, a game is said to be zero – sum game if the sum of payments to all competitors after a play of the game is restricted to be zero.

In this case the play doesn't add a single penny to the total wealth of all players; it merely results in new distribution of initial money amongst them.

Two Person Zero Sum (Or Rectangular) Game:

A game with only two players in which the gains of one player are the losses of other is called a two person zero – sum game. In other words the games in which the algebraic sum of gains and losses of all the players is zero are called zero sum games. Two person zero – sum games are also called rectangular games because these are usually represented by a pay – off matrix in rectangular form.

Pay – off Matrix:

In a two person zero sum – game, the resulting gain, can easily be represented in the form of matrix, called **pay – off matrix** or **gain matrix**. Thus a pay – off matrix is a table which shows how payments should be made at the end of the play or game.
Let us consider a game with only two players (competitors) A and B in which player A has m – courses of action and player B has n – courses of action. The game can be described by means of a pair of matrices such that:

1. Row designations for each matrix are the courses of action available to A.

2. Column designations for each matrix are the courses of action available to B.

3. The cell entries are the payments to A for the one matrix and to B for the other matrix. The cell entry a_{ij} is the payment to A in A's pay – off matrix when A chooses the course of action i and B chooses the course of action j.

4. In a zero – sum two person game, the cell entry in B's pay – off matrix will be the negative of the corresponding cell entry in A's pay – off matrix.

The pay – off matrices are as follows:

(A's pay – off matrix)

B

		1	2	…………	j	…………	n
	1	a_{11}	a_{12}	…………	a_{1j}	…………	a_{1n}
	2	a_{21}	a_{22}	…………	a_{2j}	…………	a_{2n}
A	………	…………	…………	…………		…………	…………
	i	a_{i1}	a_{i2}	…………	a_{ij}	…………	a_{in}
	………	…………	…………	…………	…………	…………	…………
	m	a_{m1}	a_{m2}	…………	a_{mj}	…………	a_{mn}

(B's pay – off matrix)

		B					
		1	2	…………	j	…………	n
	1	a_{11}	a_{12}	…………	a_{1j}	…………	a_{1n}
	2	a_{21}	a_{22}	…………	a_{2j}	…………	a_{2n}
A	………	…………	…………	…………		…………	…………
	i	a_{i1}	a_{i2}	…………	a_{ij}	…………	a_{in}
	……..	…………	…………	…………	…………	…………	…………
	m	a_{m1}	a_{m2}	…………	a_{mj}	…………	a_{mn}

For example, suppose A and B play a game, called 'Two – finger Mora'. Both simultaneously show either 1 or 2 fingers. If the number of fingers of one coincide with the number of fingers of other then the player A wins and gets Rs 1.00 from and B. If the number of fingers of the two players do not coincide then B wins and gets Rs 1.00 from A. This game is a two person zero – sum game, since the winning of one player are taken as losses of the other. For this game the pay – off matrix to A is as follows:

		B	
		1 finger	2 fingers
A	1 finger	+ 1	- 1
	2 fingers	- 1	+ 1

Strategy:

The strategy of a player is the predetermined rule by which a player decides his courses of action from his own list of courses of action during the game.

There are following two type of strategies:

Pure Strategy:

A pure strategy is a decision, in advance of all plays, always to choose a particular course of action.
A pure strategy may be identified by a number of representing the courses of action.

Mixed Strategy:

A mixed strategy is a decision, in advance of all plays, to choose a course of action for each play in accordance with some particular probability distribution.

Optimum Strategy:

A course of action or play which puts the player in the most preferred position, irrespective of strategy of his competitors, is called an optimum strategy.

The advantage of a mixed strategy over a pure strategy, after the pattern of the game become evident, is that the opponent are kept guessing as to which course of action is to be selected by the other on any particular occasion. Mathematically, a mixed strategy to any player is a set X of any m non – negative real numbers whose sum is unity. These m non – negative real numbers represent the probabilities with which each course of action (pure strategy) should be selected, m being the number of pure strategies of the player.

Thus, if x_i is the probability of choosing course i, then

$$X = (x_1, x_2, \ldots\ldots, x_m)$$

$$\text{where} \quad , x_i \geq 0, i = 1, 2, \ldots\ldots, m$$

$$\text{and}$$

$$\sum_{i=1}^{m} x_i = 1$$

If $x_i = 0$, $i \neq r$, then the mixed strategy indicate the r^{th} pure strategy. Thus, a pure strategy is a special case of a mixed strategy.

Solution of a Game:

By solving a game we mean to find the best strategies for both the players and the value of game.

The **value of the game** is a maximum generated gain to player A (maximizing player) if both the players use their strategies. It is generally denoted by **v**, and is unique. If the value of a game is zero then it is called a fair **game**.

In a game theory the best strategies for each player are determined on the basis of **maximum and minimum criterion of optimality.**

Saddle Point:

A saddle point of a pay – off matrix is that position in the matrix where the maximum of row minima coincides with the minimum of the column maxima. The cell entry (pay – off) at the saddle point is called the **value of the game**.

A game for which Maximin for A = Minimax for B is called a game with saddle point. That is in a game with saddle point the players use the pure strategies.

Solution of a Rectangular Game with Saddle Point:

The saddle point is detected as follows:

1. Select the minimum element in each row and encircle them.(□)
2. Select the maximum element in each column and enclose in small squares. (○)
2. A point which is enclosed with in the circle and square is a saddle point.

If a matrix contains more than one saddle point then there exist more than one solution of the game.

Solve the following game:

		B		
		I	II	III
A	I	6	8	6
	II	4	12	2

Solution:

		B		
		I	II	III
A	I	6	8	6
	II	4	12	2

The matrix has two saddle points (1, 1) and (1, 3). Thus, the solution of the above game is given by

1. The best strategy for player A is I.
2. The best strategy for player B is I or III.
3. Value of game is 6 for player A and – 6 for player B.

Solve the following game:

		B: I	II	III	IV	V
A	I	- 2	0	0	5	3
	II	3	2	1	2	2
	III	4	-3	0	- 2	6
	IV	5	3	- 4	2	- 6

Solution of a Rectangular Game in Terms of Mixed Strategies:

If a game does not have a saddle point, the two players can not use maximin, minimax strategies (pure) as their optional strategies, then the best strategies are mixed strategies, then the best strategies are mixed strategies. The two players, instead of selecting pure strategies only, may play their plays according to predetermined set which consist of probabilities corresponding to each of their pure strategies.

Properties of Optimal Mixed Strategies:

1. If one of the players adheres to his optimal mixed strategy and other derivatives from his optimal strategies, then the deviating players can only decreases his yield and can not increase in any case (at most may be equal).
2. If one of the players adheres to his optimal strategy then the value of the game does not alter if

Solution of 2 X 2 Game without Saddle Point:

A 2 x 2 game can be formed as follows:

		B	
		I(y_1)	II(y_2)
A	I (x_1)	a_{11}	a_{12}
	II(x_2)	a_{21}	a_{22}

By solving this game algebraically for the strategies of A and B we get:

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$x_2 = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y_2 = \frac{a_{11} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

and ,

$$v = \frac{a_{11} a_{22} - a_{12} a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Where,

x_1 = The probability that player A chooses strategy I

x_2 = The probability that player A chooses strategy II

y_1 = The probability that player B chooses strategy I

y_2 = The probability that player B chooses strategy II

Solve the game who’s pay – off matrix is given by:

$$\begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix}$$

Solution:

First we try to find the saddle point for the game by the rule described earlier and get the following table

$$\begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix}$$

Rule of Dominance

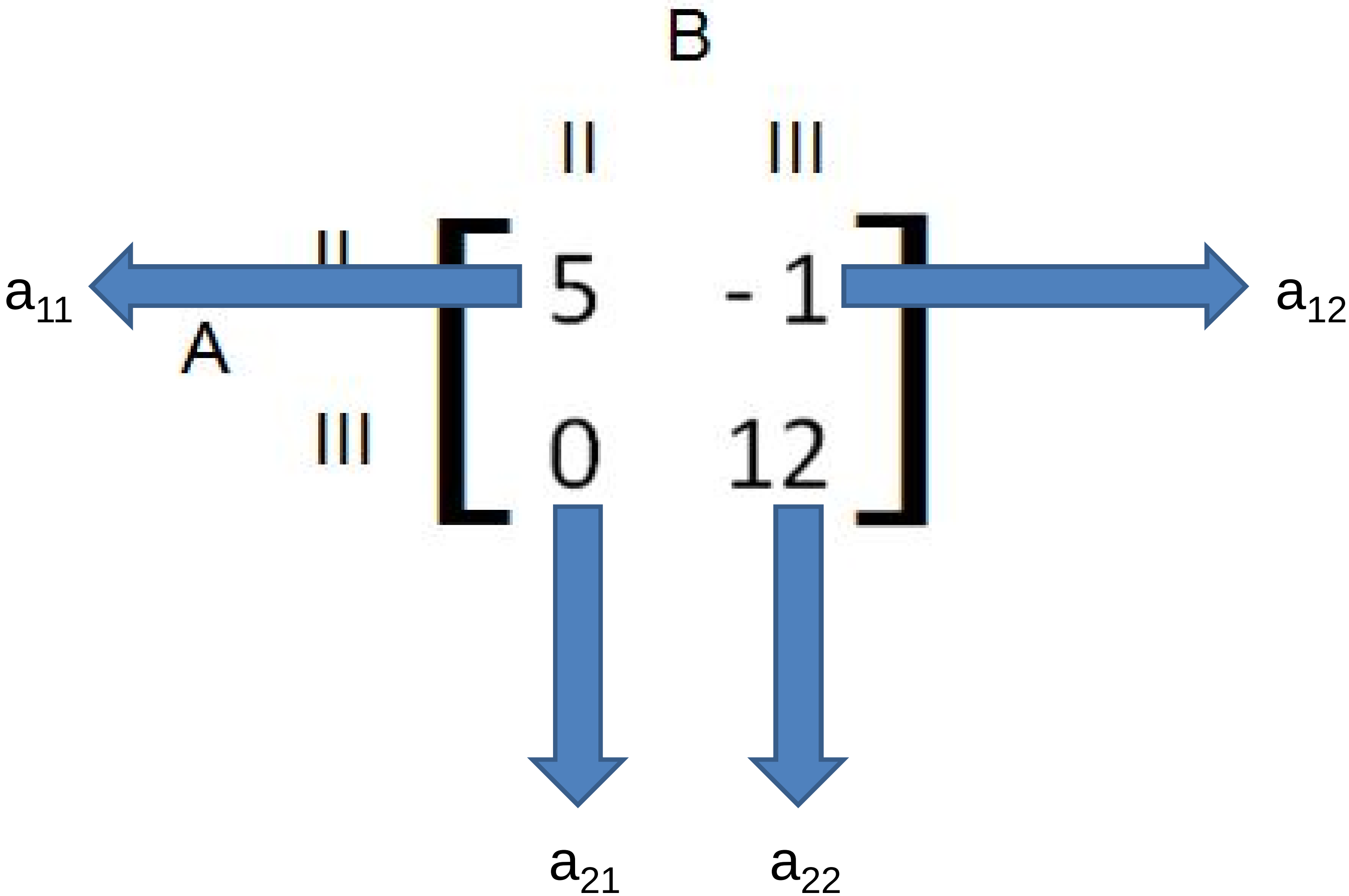
Graphical Method

B

		I	II	III
[redacted]				
A	II	7	5	- 1
	III	6	0	12

[redacted]

		B	
		II	III
A	II	5	-1
	III	0	12



B
II
III
II
A
III
5
- 1
0
12
a11
a12
a21
a22

$$x_1 = \frac{2}{3}, x_2 = \frac{1}{3}, y_1 = \frac{13}{18}, y_2 = \frac{5}{18}, v = \frac{10}{3}$$

Best strategy for player A is given by; $\left(0, \frac{2}{3}, \frac{1}{3}\right)$

Best strategy for player B is given by; $\left(0, \frac{13}{18}, \frac{5}{18}\right)$

Value of game is given by; $v = \frac{10}{3}$

Graphical Solution of 2 × n and m × 2 games:

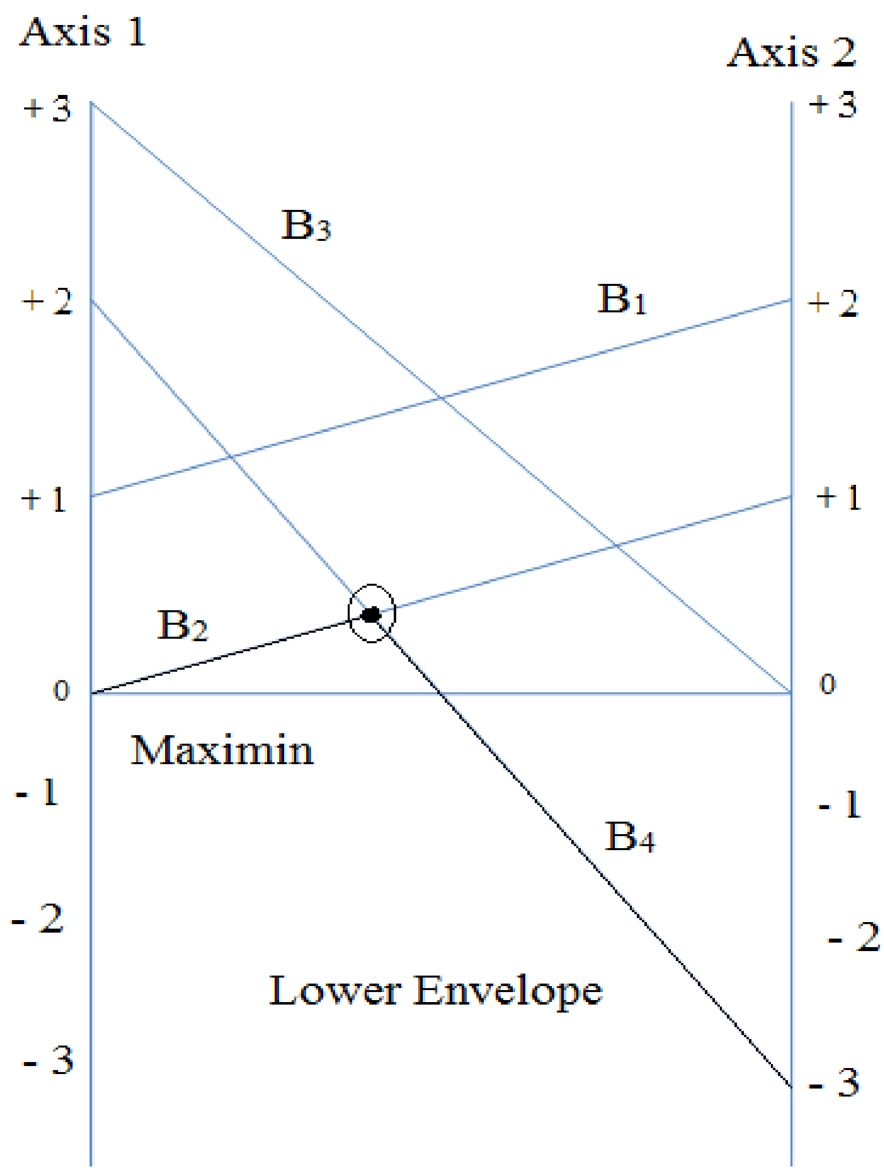
Solve the following 2 × 4 game:

		Player B			
		I	II	III	IV
Player A	I	2	1	0	- 2
	II	1	0	3	2

The problem doesn’t possess a saddle point

A’s expected pay off’s against B’s pure moves are given by;

B’s Pure Strategy	A’s expected payoff $E(P_1)$
B_1	$E_1(P_1) = 2P_1 + P_2 = P_1 + 1$
B_2	$E_2(P_1) = P_1 + 0P_2 = P_1$
B_3	$E_3(P_1) = 0P_1 + 3P_2 = -3P_1 + 3$
B_4	$E_4(P_1) = -2P_1 + 2P_2 = -4P_1 + 2$


Axis 1
Axis 2
+ 3
+ 3
B3
B1
+ 2
+ 2
+ 1
+ 1
B2
0
0
Maximin
- 1
- 1
B4
- 2
- 2
Lower Envelope
- 3
- 3